The Role of the Korteweg–de Vries Hierarchy in the N–Soliton Dynamics of the Shallow Water Wave Equation

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Abstract

We apply a multiple–time version of the reductive perturbation method to study long waves as governed by the shallow water wave model equation. As a consequence of the requirement of a secularity–free perturbation theory, we show that the well known N–soliton dynamics of the shallow water wave equation, in the particular case of $\alpha = 2\beta$, can be reduced to the N–soliton solution that satisfies simultaneously all equations of the Korteweg–de Vries hierarchy.

Keywords: Perturbation Theory, Secularity, KdV Hierarchy

I. INTRODUCTION

The shallow water wave (SWW) equation,

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0 \,, \tag{1}$$

is known to present solitonic solutions for two families of values of the parameters α and β. The first family, given by

$$\alpha = 2\beta \,, \tag{2}$$

was shown to be integrable by the inverse scattering method by Ablowitz, Kaup, Newell and Segur.[1] The second family, given by

$$\alpha = \beta \,, \tag{3}$$

was shown to possess N soliton solutions by Hirota and Satsuma.[2] In addition, it has been shown[3,4] that Eq.(1) passes the Painlevé test if and only if either condition (2) or (3) is satisfied, which is an indication that this equation might be integrable only for these two cases. We would like also to remark that Eq.(1) may be written as an integrodifferential equation:

$$U_{xxt} + \alpha U U_t - \beta U_x \int_x^\infty U_t dx' - U_t - U_x = 0 \,. \tag{4}$$

In this form, the SWW equation was shown to be derivable in the Boussinesq approximation from the classical shallow water theory,[4] but with the condition

$$\beta = 2\alpha \,, \tag{5}$$

which is neither of the above discussed cases.

In this paper, we are going to apply a multiple–time[5,6] reductive perturbation method[7] to study the long–wave limit of the SWW equation. This method, when applied to non-linear weak dispersive general systems, introduces a connection between the equations of

the Korteweg–de Vries (KdV) hierarchy and the elimination of the soliton–related secularities appearing in the evolution equations of the higher–order terms of the wave–field. As a consequence of this approach, we will show that the well known solitary–wave solution to Eq.(1), existing for any α and β, can be written as a solitary–wave satisfying simultaneously all equations of the KdV hierarchy. This result is a consequence of an intrinsic property of the KdV solitary–wave, which allows for a truncation of the perturbative series. Then, we will show that, for the case $\alpha = 2\beta$, the N soliton dynamics of the SWW equation can be reduced to the dynamics of the N soliton solution of the KdV hierarchy. This result[8] comes from the fact that the SWW equation with $\alpha = 2\beta$, and the KdV equation are both related to the same eigenvalue problem in the AKNS scheme.[1] On the other hand, for $\alpha = \beta$, the dynamics underlying the N soliton solutions of the SWW equation has a different nature, being similar to the dynamics governing the N soliton solution of the Boussinesq equation.[3] This case will not be considered in this paper, which is organized as follows. In Section II we introduce the multiple–time perturbation scheme based on the long–wave expansion of the dispersion relation. In Section III we obtain the first few evolution equations, and in Section IV we show how the KdV hierarchy equations appear as a compatibility condition. The case of a solitary–wave solution for the KdV hierarchy is considered in Section V, and in Section VI we describe how the N soliton solution of the SWW equation with $\alpha = 2\beta$ can be obtained as the N soliton solution of the KdV hierarchy. And finally, in Section VII we summarize and discuss the results obtained.

II. LONG–WAVE EXPANSION AND THE MULTIPLE TIMES

The linear dispersion relation of the SWW equation is given by

$$\omega(k) = \frac{k}{1 + k^2}\,. \tag{6}$$

To study its long–wave limit, we put

$$k = \epsilon\kappa\,, \tag{7}$$

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with ϵ a small parameter, and we expand the dispersion relation according to

$$\omega(\kappa) = \epsilon\kappa - \epsilon^3\kappa^3 + \epsilon^5\kappa^5 - \epsilon^7\kappa^7 + \cdots . \tag{8}$$

Introducing this expansion into the plane wave solution of the linear SWW equation, we get

$$u = a \exp i \left[\kappa\epsilon(x - t) + \kappa^3\epsilon^3 t - \kappa^5\epsilon^5 t + \kappa^7\epsilon^7 t - \cdots \right] , \tag{9}$$

where a is a constant. Inspired by this solution, we introduce a slow space

$$\xi = \epsilon(x - t) , \tag{10}$$

as well as an infinite sequence of properly normalized slow time variables

$$\tau_3 = \epsilon^3 t \quad ; \quad \tau_5 = -\epsilon^5 t \quad ; \quad \tau_7 = \epsilon^7 t \quad ; \quad \text{etc.} \tag{11}$$

Accordingly, we have that

$$\frac{\partial}{\partial x} = \epsilon\frac{\partial}{\partial \xi} \tag{12}$$

and

$$\frac{\partial}{\partial t} = -\epsilon\frac{\partial}{\partial \xi} + \epsilon^3\frac{\partial}{\partial \tau_3} - \epsilon^5\frac{\partial}{\partial \tau_5} + \epsilon^7\frac{\partial}{\partial \tau_7} - \cdots . \tag{13}$$

As we are going to see later, the slow time normalizations introduced in the definitions (11) are crucial in the sense that they will allow for an automatic elimination of the secularities appearing in the evolution equations of the higher order terms of the wave–field.[9]

III. MULTIPLE TIME EVOLUTION EQUATIONS

We expand now the wave–field u according to

$$u = \epsilon\hat{u} = \epsilon \left(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \right) , \tag{14}$$

and substitute it, together with Eqs.(12) and (13), in the SWW equation (1). The result is:

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$$\frac{\partial^3}{\partial \xi^3}\Big(-\epsilon \frac{\partial}{\partial \xi} + \epsilon^3 \frac{\partial}{\partial \tau_3} - \epsilon^5 \frac{\partial}{\partial \tau_5} + \cdots\Big)\hat{u} - \frac{\partial}{\partial \xi}\Big(\epsilon \frac{\partial}{\partial \tau_3} - \epsilon^3 \frac{\partial}{\partial \tau_5} + \cdots\Big)\hat{u}$$

$$+ \alpha \frac{\partial \hat{u}}{\partial \xi}\Big(-\epsilon \frac{\partial^2}{\partial \xi^2} + \epsilon^3 \frac{\partial}{\partial \xi}\frac{\partial}{\partial \tau_3} - \epsilon^5 \frac{\partial}{\partial \xi}\frac{\partial}{\partial \tau_5} + \cdots\Big)\hat{u}$$

$$+ \beta \frac{\partial^2 \hat{u}}{\partial \xi^2}\Big(-\epsilon \frac{\partial}{\partial \xi} + \epsilon^3 \frac{\partial}{\partial \tau_3} - \epsilon^5 \frac{\partial}{\partial \tau_5} + \cdots\Big)\hat{u} = 0 \,. \tag{15}$$

We proceed then to an order–by–order inspection of this equation. At order ϵ^0 we get

$$\frac{\partial^4 u_0}{\partial \xi^4} + \frac{\partial^2 u_0}{\partial \xi \partial \tau_3} + (\alpha + \beta)\frac{\partial u_0}{\partial \xi}\frac{\partial^2 u_0}{\partial \xi^2} = 0 \,. \tag{16}$$

We make now, in all components u_n of the wave–field expansion, the following transformation:

$$\frac{\partial u_n}{\partial \xi} = -\frac{6}{\alpha + \beta}\, v_n \quad ; \quad n = 0, 1, 2, \ldots \;\;. \tag{17}$$

Consequently, Eq.(16) acquires the form

$$\frac{\partial v_0}{\partial \tau_3} + \frac{\partial^3 v_0}{\partial \xi^3} - 6 v_0 \frac{\partial v_0}{\partial \xi} = 0 \,, \tag{18}$$

which is the KdV equation in the time τ_3.

At order ϵ^1, Eq.(15) gives:

$$\frac{\partial^4 u_1}{\partial \xi^4} + \frac{\partial^2 u_1}{\partial \xi \partial \tau_3} + (\alpha + \beta)\frac{\partial u_0}{\partial \xi}\frac{\partial^2 u_1}{\partial \xi^2} + (\alpha + \beta)\frac{\partial u_1}{\partial \xi}\frac{\partial^2 u_0}{\partial \xi^2} = 0 \,. \tag{19}$$

Transforming u_0 and u_1 according to Eq.(17), we obtain

$$\frac{\partial v_1}{\partial \tau_3} + \frac{\partial^3 v_1}{\partial \xi^3} - 6\frac{\partial}{\partial \xi}(v_0 v_1) = 0 \,, \tag{20}$$

that is, v_1 satisfies a homogeneous linearized KdV equation in the time τ_3.

At order ϵ^2, Eq.(15) gives:

$$-\frac{\partial^4 u_2}{\partial \xi^4} + \frac{\partial^4 u_0}{\partial \xi^3 \partial \tau_3} - \frac{\partial^2 u_2}{\partial \xi \partial \tau_3} + \frac{\partial^2 u_0}{\partial \xi \partial \tau_5}$$

$$-(\alpha + \beta)\left(\frac{\partial u_0}{\partial \xi}\frac{\partial^2 u_2}{\partial \xi^2} + \frac{\partial u_1}{\partial \xi}\frac{\partial^2 u_1}{\partial \xi^2} + \frac{\partial u_2}{\partial \xi}\frac{\partial^2 u_0}{\partial \xi^2}\right) + \alpha \frac{\partial u_0}{\partial \xi}\frac{\partial^2 u_0}{\partial \xi \partial \tau_3} + \beta \frac{\partial^2 u_0}{\partial \xi^2}\frac{\partial u_0}{\partial \tau_3} = 0 \,. \tag{21}$$

Transforming u_0, u_1 and u_2 according to Eq.(17), and using the KdV equation (18) to express $v_{0\tau_3}$, we obtain

$$\frac{\partial v_2}{\partial \tau_3} + \frac{\partial^3 v_2}{\partial \xi^3} - 6\frac{\partial}{\partial \xi}(v_0 v_2) = \frac{\partial v_0}{\partial \tau_5} + \left[\frac{18\alpha + 24\beta}{\alpha + \beta}\right]\frac{\partial v_0}{\partial \xi}\frac{\partial^2 v_0}{\partial \xi^2}$$
$$+ \left[\frac{12\alpha + 6\beta}{\alpha + \beta}\right]v_0\frac{\partial^3 v_0}{\partial \xi^3} - \left[\frac{36\alpha + 18\beta}{\alpha + \beta}\right]v_0{}^2\frac{\partial v_0}{\partial \xi} - \frac{\partial^5 v_0}{\partial \xi^5} + 6v_1\frac{\partial v_1}{\partial \xi} \ . \tag{22}$$

This equation involves the evolution of v_0 in the time τ_5, which is not known up to this point. Moreover, when v_0 is assumed to be a solitary–wave solution of the KdV equation, the source term $\partial^5 v_0/\partial \xi^5$ will be a secular producing term. Therefore, before continuing we have to solve these two problems.

IV. THE KDV HIERARCHY AS A COMPATIBILITY CONDITION

As we have already seen, v_0 satisfies the KdV equation in the time τ_3. The evolution of v_0 in the higher order times τ_{2n+1} can then be obtained in the following way.[5] First, to have a well ordered perturbative scheme, we impose that each one of these equations be ϵ–independent when passing from the slow (v_0, ξ, τ_{2n+1}) to the laboratory coordinates (v, x, t). This will select all possible terms to appear in the equation for $v_{0\tau_{2n+1}}$. Then, by imposing the natural (in the multiple time formalism) compatibility condition

$$\left(v_{0\tau_3}\right)_{\tau_{2n+1}} = \left(v_{0\tau_{2n+1}}\right)_{\tau_3}, \tag{23}$$

it is possible to determine the above constants in terms of α_{2n+1}, which is left as a free-parameter. As it can be easily verified through an explicit calculation,[5] the resulting equations are exactly those given by the KdV hierarchy. As an example, let us consider the evolution of v_0 in time τ_5. From the ϵ-independence requirement, $v_{0\tau_5}$ is restricted to be of the form

$$v_{0\tau_5} = \alpha_5 v_{0(5\xi)} + \beta_5 v_0 v_{0\xi\xi\xi} + \gamma_5 v_{0\xi} v_{0\xi\xi} + \delta_5 v_0^2 v_{0\xi}, \tag{24}$$

where α_5, β_5, γ_5 and δ_5 are constants. The compatibility condition

$$\left(v_{0\tau_3}\right)_{\tau_5} = \left(v_{0\tau_5}\right)_{\tau_3},$$

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then, determines β_5, γ_5 and δ_5 in terms of α_5, which is left as a free-parameter, yielding the equation

$$v_{0\tau_5} = v_{0(5\xi)} - 10v_0 v_{0\xi\xi\xi} - 20v_{0\xi} v_{0\xi\xi} + 30{v_0}^2 v_{0\xi} . \tag{25}$$

Analogously, for the evolution of v_0 in τ_7, we obtain

$$v_{0\tau_7} = -v_{0(7\xi)} + 14v_0 v_{0(5\xi)} + 42v_{0\xi} v_{0(4\xi)} + 140(v_0)^3 v_{0\xi}$$
$$+ 70v_{0\xi\xi} v_{0\xi\xi\xi} - 280v_0 v_{0\xi} v_{0\xi\xi} - 70(v_{0\xi})^3 - 70{v_0}^2 v_{0\xi\xi\xi} . \tag{26}$$

Equations (25) and (26) are the first two higher–order equations of the KdV hierarchy. The same procedure can be used to generate any higher-order equation of the KdV hierarchy. The right–hand side of these equations would in principle appear multiplied by the corresponding free–parameter left at each order, which would account for different possible slow time normalizations. However, since we have already defined appropriate slow time normalizations, these parameters were taken to be 1 in order to have an agreement with the normalizations introduced in Eq.(11). This is an important point since it will allow for an automatic elimination[9] of the soliton related secular producing terms present in the higher-order perturbation equations, which are always of the form[10]

$$v_{0(2n+1)\xi} \quad ; \quad n = 0, 1, 2, \dots . \tag{27}$$

V. THE SOLITARY–WAVE SOLUTION

In this section, we are going to consider some specific solutions to our equations. First, we will assume for v_1 the trivial solution

$$v_1 = 0 . \tag{28}$$

Then, by taking into account that the evolution of v_0 in the time τ_5 is now known to proceed according to the fifth–order equation of the KdV hierarchy, given by Eq.(25), we obtain for Eq.(22)

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$$v_{2\tau_3} + v_{2\xi\xi\xi} - 6(v_0 v_2)_\xi = \frac{2\alpha - 4\beta}{\alpha + \beta} \left[-3v_0{}^2 v_{0\xi} + v_0 v_{0\xi\xi\xi} - v_{0\xi} v_{0\xi\xi} \right] . \tag{29}$$

We see in this way that the use of the properly normalized KdV hierarchy equation to express $v_{0\tau_5}$ has automatically canceled out the secular producing term $v_{0(5\xi)}$ from Eq.(22). Therefore, besides satisfying the KdV equation in the time τ_3, the wave–field v_0 must satisfy also the fifth–order KdV hierarchy equation (25) in the time τ_5. Actually, in order to have a secularity–free perturbation theory up to any higher order, v_0 will be required to satisfy simultaneously all equations of the KdV hierarchy, each one in a different slow time coordinate.

Next, we assume for v_0 a KdV solitary–wave solution. As v_0 must satisfy simultaneously all equations of the KdV hierarchy, we take

$$v_0 = -2\kappa^2 \mathrm{sech}^2 \left[\kappa\xi - 4\kappa^3 \tau_3 + 16\kappa^5 \tau_5 - 64\kappa^7 \tau_7 + \cdots \right] . \tag{30}$$

Now, using this solitary–wave solution, one can easily check that the right–hand side of Eq.(29) vanishes, implying that, along with v_1, v_2 also satisfies a homogeneous linearized KdV equation:

$$v_{2\tau_3} + v_{2\xi\xi\xi} - 6(v_0 v_2)_\xi = 0 . \tag{31}$$

Like in the case of v_1, we will assume for it the trivial solution

$$v_2 = 0 . \tag{32}$$

We proceed now to the next order, in which case Eq.(15) gives

$$u_{3(4\xi)} - u_{3\xi\tau_3} + (\alpha + \beta) \left(u_{0\xi} u_{3\xi} \right)_\xi = 0 , \tag{33}$$

where, due to the fact that $v_1 = v_2 = 0$, we have already used that $u_1 = u_2 = 0$. Then, by transforming u_0 and u_3 according to Eq.(17), we get the following equation for v_3:

$$v_{3\tau_3} + v_{3\xi\xi\xi} - 6 \left(v_0 v_3 \right)_\xi = 0 . \tag{34}$$

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We see in this way that also v_3 satisfies a homogeneous linearized KdV equation, and as in the previous cases we choose for it the trivial solution

$$v_3 = 0 \,. \tag{35}$$

This is actually what will happen at every higher order: after using the equations of the KdV hierarchy to express $v_{0\tau_{2n+1}}$, and after assuming the solitary–wave solution (30) for v_0, the evolution of v_n in the time τ_3 will be given by a homogeneous linearized KdV equation. Consequently, it will always be possible to choose the trivial solution

$$v_n = 0 \quad ; \quad n = 1, 2, 3, \ldots \,. \tag{36}$$

It should be mentioned that the evolution of v_n $(n = 1, 2, 3, \ldots)$ in each higher–order time τ_{2n+3} is always given by a *linear* partial differential equation, which becomes homogeneous after the choice $v_{n-1} = 0$. Therefore, the solution (36) is in indeed always possible and compatible with the whole perturbative scheme.

We return now to the laboratory coordinates (x, t, u). First, we recall that we have assumed for u the expansion given by Eq.(14). Moreover, we have obtained a particular solution in which $u_1 = u_2 = u_3 = \cdots = 0$. Therefore, the laboratory wave–field u is given simply by

$$u = \epsilon u_0 \,. \tag{37}$$

On the other hand, v_0 was taken to be given by Eq.(30), which is a solitary–wave satisfying all equations of the KdV hierarchy. By using the transformation (17), we can obtain the corresponding u_0:

$$u_0 = -\frac{12\kappa^2}{\alpha + \beta} \int_\xi^\infty \mathrm{sech}^2 \left[\kappa\xi' - 4\kappa^3\tau_3 + 16\kappa^5\tau_5 - 64\kappa^7\tau_7 + \cdots \right] d\xi' \,. \tag{38}$$

Integrating and using Eq.(37), we get

$$u = -\frac{12\epsilon\kappa}{\alpha + \beta} \left[1 - \tanh(\kappa\xi - 4\kappa^3\tau_3 + 16\kappa^5\tau_5 - 64\kappa^7\tau_7 + \cdots) \right] \,. \tag{39}$$

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We drive now u back to the laboratory by eliminating the slow variables $(\kappa, \xi, \tau_{2n+1})$, which are related to the laboratory ones (k, x, t) according to equations (7), (10) and (11). The result is

$$u = -\frac{12k}{\alpha + \beta}\left[1 - \tanh\left(kx - k\{1 + 4k^2 + 16k^4 + 64k^6 + \cdots\}t\right)\right] . \qquad (40)$$

But, the series appearing inside the curly brackets can be summed, yielding finally

$$u = -\frac{12k}{\alpha + \beta}\left[1 - \tanh\left(kx - \frac{kt}{1 - 4k^2}\right)\right] , \qquad (41)$$

which is the well known solitary–wave solution of the SWW equation (1), valid for any value of α and β, except of course $\alpha = -\beta$.

VI. TWO–OR–MORE SOLITON SOLUTIONS

The elimination of secular producing terms through the multiple–scale method is not always possible in the case where a N–soliton solution is assumed for the KdV equation because of the possible appearance of obstacles in the perturbative series.[11] Obstacles are higher–order effects coming from the nonintegrability of the original equation which, when present, preclude the existence of uniform asymptotic expansions. An example of such a manifestation can be found in Ref.[12]. Here, however, we consider a particular case where no obstacles are present, namely the SWW equation with $\alpha = 2\beta$.

Let us then return to the perturbation theory, but considering now , the case in which v_0 is assumed to be the two–soliton solution of the KdV equation (18), which is given by

$$v_0 = -4(\kappa_2^2 - \kappa_1^2)\left[\frac{\kappa_2^2 - \kappa_1^2 + \kappa_1^2 \cosh(2\kappa_2\gamma_2) + \kappa_2^2 \cosh(2\kappa_1\gamma_1)}{\left[(\kappa_2 - \kappa_1)\cosh(\kappa_1\gamma_1 + \kappa_2\gamma_2) + (\kappa_2 + \kappa_1)\cosh(\kappa_1\gamma_1 - \kappa_2\gamma_2)\right]^2}\right] , \qquad (42)$$

with

$$\gamma_i = \xi - 4\kappa_i^2\tau_3 \ , i = 1, 2 . \qquad (43)$$

In this case, the whole perturbative scheme follows closely the one when v_0 is a one–soliton solution to the KdV equation. In particular, provided the evolution of v_0 in the time τ_5 be

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given by the fifth–order KdV hierarchy equation (25), the equation for v_2 in the time τ_3 will again be given by

$$v_{2\tau_3} + v_{2\xi\xi\xi} - 6(v_0 v_2)_\xi = \frac{2\alpha - 4\beta}{\alpha + \beta}\left[-3{v_0}^2 v_{0\xi} + v_0 v_{0\xi\xi\xi} - v_{0\xi} v_{0\xi\xi}\right] . \tag{44}$$

However, when the two–soliton solution (42) is substituted in its right–hand side, it does not vanish as it did when v_0 was assumed to be the solitary–wave solution (30). But, as can be easily seen, for the specific case of $\alpha = 2\beta$, Eq.(44) will be homogeneous independently of the solution chosen for v_0. Therefore, also for the case in which v_0 represents a two–soliton solution, it is possible to choose the trivial solution $v_2 = 0$. This is actually what happens at any higher order. In fact, to obtain a secularity–free perturbation theory, we first notice that v_0 must satisfy simultaneously all equations of the KdV hierarchy, which means that instead of (43) we have ($i = 1, 2$):

$$\gamma_i = \xi - 4\kappa_i^2 \tau_3 + 16\kappa_i^4 \tau_5 - 64\kappa_i^6 \tau_7 + \cdots . \tag{45}$$

Then, for the case $\alpha = 2\beta$, the evolution of v_n in the time τ_3 will always be given by a homogeneous linearized KdV equation, whose solution we assume to be

$$v_n = 0 \quad ; \quad n = 1, 2, 3, \ldots .$$

In other words, the perturbative series truncates, and an exact solution $u = \epsilon^2 u_0$ can be found also for a two–soliton solution.

The return to the laboratory coordinates (x, t, u) proceeds in the very same way as for the case of the solitary–wave. Writing ($i = 1, 2$)

$$\kappa_i = \epsilon^{-1} k_i ,$$

and using Eqs.(10) and (11), we find that, in the laboratory coordinates, the two soliton solution of the KdV hierarchy is given by

$$u = \epsilon u_0 = -\frac{12}{\alpha + \beta}(k_1 + k_2)\left[1 + \frac{(k_1 - k_2)\tanh(k_2\gamma_2)}{k_2 - k_1 \tanh(k_1\gamma_1)\tanh(k_2\gamma_2)}\right] , \tag{46}$$

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where now

$$\gamma_i = x - \frac{t}{1 - 4k_i^2} \quad i = 1, 2\,.$$

(47)

This is exactly the two–soliton solution of the SWW equation when $\alpha = 2\beta$. Therefore, like in the case of a solitary–wave, the two–soliton solution of the SWW equation is nothing but a two–soliton satisfying simultaneously, in the slow variables, all equations of the KdV hierarchy. Actually, as a similar procedure shows, this statement holds for the N–soliton solution as well.

The above results indicate that there exists a close relation between the KdV hierarchy and the SWW equation with $\alpha = 2\beta$. The root of this relation becomes more explicit in the AKNS scheme.[1] Indeed, as a by–product of this scheme, it has been shown that the family of equations

$$q_t + \hat{C}(4L)q_x = 0\,,$$

(48)

where

$$L = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}q_x \int_x^\infty dy\,,$$

(49)

and $\hat{C}(k^2) = \omega(k)/k$, with $\omega(k)$ a given dispersion relation, is integrable, having all of them the same associated eigenvalue problem. In the specific case of $\hat{C}(k^2) = 1/(1 + k^2)$, Eq.(48) is just the SWW equation with $\alpha = 2\beta$. On the other hand, if we expand $\hat{C}(k^2)$ according to

$$\hat{C}(k^2) = 1 - k^2 + k^4 - k^6 + \cdots\,,$$

(50)

we get

$$q_t + q_x - 4L(q_x) + 16L^2(q_x) - \cdots = 0\,.$$

(51)

When passing to the multiple–time slow variables (q_0, ξ, τ_{2n+1}), with $q_0 = \epsilon^{-2}q$, one can immediately see the emergence of the KdV hierarchy as

$$q_{0\tau_{2n+1}} = (4L)^n q_{0\xi}$$

does generate the hierarchy.[13)] It should be mentioned that the above result is independent of any specific solution to the KdV hierarchy.

VII. FINAL REMARKS

We have in this paper suceeded in writing the solitary–wave solution of the SWW equation (1), valid for any value of the parameters α and β, as a solitary–wave satisfying simultaneously, in the slow variables, all equations of the KdV hierarchy. This is actually a result valid for any system presenting an exact solitary–wave solution, that is, for systems whose solitary–wave initial condition does not radiate. It does not matter, in this case, whether the system is integrable or not.

We have then shown that, when v_0 is assumed to be not a solitary–wave but a two or more soliton solution of the KdV hierarchy, the right–hand side of Eq.(29), as well as all other corresponding higher order equations, does not vanish anymore, which in principle would mean that the perturbative series does not truncate. However, in the particular case of $\alpha = 2\beta$, all those equations become homogeneous again, making then possible the truncation of the perturbative series through the choice of the trivial solutions

$$v_n = 0 \quad ; \quad n = 2, 3, \ldots . \tag{52}$$

The return to the laboratory coordinates leads v_0 to the two–or–more soliton solution of the SWW equation with $\alpha = 2\beta$, which ultimately means that the N–soliton solution of this SWW equation is nothing but a N–soliton solution satisfying simultaneously all equations of the KdV hierarchy. We have in this way shown that the dynamics underlying such solutions can be given in terms of that governing the N–soliton solutions of the whole KdV hierarchy, a fact that can also be seen to be true through the AKNS scheme.

It is important to notice the difference between the cases $N = 1$ and $N \geq 2$. For $N = 1$, the dynamics are equivalent for any α and β. However, for $N \geq 2$, the above equivalence exists only for $\alpha = 2\beta$. The reason for this difference is the presence of soliton interactions

in the case $N \geq 2$. As is well known, the SWW equation is in general non–integrable, which means that soliton interaction is inelastic. And it is exactly this inelastic character of the soliton interaction that will generate secularities other than those originated from the linear terms of the higher–order equations. What happens in the multiple scale method is that these new secularities, originated from the non–integrability of the original system, can not eliminated by the integrable equations of the KdV hierarchy. In this sense, the case with $\alpha = 2\beta$ is indeed peculiar, and can not be extended to the general case with any α and β. In other words, being the original system non–integrable, it can not be described by the integrable system formed by the equations of the KdV hierarchy.

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